MATH.APP.720

Optimization Methods — Exam, 11.12.2019 — Henri Hansen

Please answer **four** out of the following five questions, on a separate paper. Note that there are two pages in the exam. No books. *non-programmable calculators are allowed.*
A Collection of Formulas will be handed. Kaavakokoelma jaetaan.

1. Consider the optimization problem:

$$\min f(x,y) = x^2 + y^3$$

s.t.

$$x^2 + (y-2)^4 \leq 1$$

   (a) Is the feasible set convex?

   (b) Is the cost-function convex in the feasible region?

   (c) You can deduce the optimum of this problem relatively easily. Demonstrate that it is a KKT-point and check the constraint qualifications.

2. Assume that we have the following linear problem:

$$\min c^T x$$

s.t.

$$Ax \leq b$$
$$x \geq 0$$

   (a) Explain how this is transformed so that the constraint becomes an equality constraint.

   (b) Suppose that we wish to use the Simplex algorithm. To do so, we need one basic feasible solution to the LP. Explain how to find one, assuming b above contains only positive values.

   (c) Consider the reduced cost $\hat{c}_N$ (formula given in the formula sheet). Show (using algebra), that its component i is equal to the change in the cost function if we increase the value of the null-variable x_i by one.

3. Consider the unconstrained problem

$$\min f(x,y) = (1-x)^2 + 10(y-x^2)^2$$

   (a) Find the minimum of this function analytically. (Hint: It is not hard, if you really look at when it can be minimum. Or solve for the zero of the gradient, but that is going to be hard.

   (b) Consider finding the minimum using one of the methods. Start from any point (other than the optimum) and calculate one iteration. If you use the gradient method, be sure to formulate the minimization problem solving the step length.

(c) Demonstrate that the function is not convex. (By any means you like.)

4. Let $f, g : X \to \mathbb{R}$ be two strictly convex functions. Which of the following claims hold? Prove or disprove (i.e., give an example where the claim is false)

   (a) The function $\min\{f(x), g(x)\}$ is convex.

   (b) The set $f(x) = b$ is convex for every $b \in \mathbb{R}$.

   (c) If f is continuosly differentiable and $\lim_{||x|| \to \infty} f(x) = \infty$ Then f has a unique minimum.

5. Suppose we have a constraint problem

$$\min x^2 y^2$$

s.t.

$$x^2 - y^2 = 4$$

   (a) Formulate the KKT-conditions for this problem.

   (b) solve the problem either by solving the KKT-equation or by using Lagrange multipliers. (Hint: it's not too hard, consider the factors)

   (c) What is the minimum of the function?

Algorithm	direction	line length	Update rule
Gradient Descent	$-\nabla f(x_k)$	$\min_t f(x_k + td_k)$	$x_{k+1} = x_k + td_k$
Newton Method	$-f''(x_k)^{-1}\nabla f(x_k)$	1	$x_{k+1} + x_k - f''(x_k)^{-1}\nabla f(x_k)$
Quasi-Newton	$-H_k^{-1}\nabla f(x_k)$	$\min_t f(x_k + td_k)$	$x_{k+1} = x_k + td_k$ $H_{k+1} = H_k + \Delta_k$
Conjugate Gradient	$d_0 = -\nabla f(x_0)$ $d_{k+1} = -\nabla f(x_{k+1}) + \beta_k d_k$	$t_k = \frac{\|\|\nabla f(x_k)\|\|^2}{d_k^T H(x_k) d_k}$	$x_{k+1} = x_k + t_k d_k$ $\beta_k = \frac{\nabla f(x_{k+1})^T(\nabla f(x_{k+1}) - \nabla f(x_k))}{\|\|\nabla f(x_k)^2\|\|}$

1. The Subset $K \subseteq X$ of a vector space X is *convex* if and only if, for all $x, y \in K$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in K$.

2. Let K be a convex set. A function $f : K \to \mathbb{R}$ is *convex* if and only if for all $x, y \in K$ and $\lambda \in [0,1]$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

3. Let K be a convex set. A differentiable function $f : K \to \mathbb{R}$ is *pseudoconvex* if and only if for all $x, y \in K$, if $\nabla f(x) \cdot (y - x) \geq 0$ then $f(y) \geq f(x)$.

4. Let K be a convex set. A function $f : K \to \mathbb{R}$ is *quasiconvex* if and only if for all $x, y \in K$ and $\lambda \in [0,1]$, $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$

Consider the optimization problem:

$$\min f(x)$$

s.t.

$$g_i(x) \leq 0 \quad \text{where } i = 1, \ldots, m$$
$$h_j(x) = 0 \quad \text{where } j = 1, \ldots, p$$

where $f : X \to \mathbb{R}$ and $g_i : X \to \mathbb{R}$ $h_j : X \to \mathbb{R}$ are functions.

1. The Karush-Kuhn-Tucker Conditions are satisfied at point $x^* \in X$ (KKT-point) if and only if there exists $\mu_i \geq 0$ and $\lambda_j \in \mathbb{R}$ such that

$$\nabla f(x^*) + \sum_{i=i}^{m} \mu_i \nabla g_i(x^*) + \sum_{j=1}^{p} \lambda h_j(x^*) = 0$$

$$\mu_i g_i(x^*) = 0, \text{ for all } i = 1, \ldots, m$$

$$g_i(x^*) \leq 0 \text{ and } h_j(x^*) = 0 \text{ for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, p$$

Theorem: (Regarding constraint qualifications). If x^* is feasible, minimum point, then it is a KKT-point, provided one of the following hold:

1. The functions g_i are convex, the functions h_j are affine (linear) and there exists an interior point $\hat{x}$, i.e., such that $g(\hat{x}) < 0$ and $h(\hat{x}) = 0$.
2. The gradients $\nabla g_i(x^*)$ and $\nabla h_j(x^*)$ are linearly independent, for every i such that $g_i(x^*) = 0$.

(If some point does not satisfy the second qualification, it might be a minimum even if it is not a KKT point; if the first holds for the problem, it should not have points that do not satisfy the qualification)

Consider the LP-problem

$$\min c^T x$$

s.t.

$$Ax = b$$

$$x \geq 0$$

1. If $A = [B\ N]$ and $x = [x_b^T\ x_n^T]^T$ where $x_n = 0$ and $x_b \geq 0$, then we say that x is a basic feasible solution and B is a *basis* of the problem
2. Given a basis B (i.e. $A = [B\ N]$) and basic feasible solution $x = [x_b^T\ x_n^T]^T$ where $x_b > 0$, the *reduced cost* associated with B is

   $$\hat{c}_N^T = c_N^T - c_B^T B^{-1} N$$

   where $c = [c_B^T\ c_N^T]^T$.
3. The *dual* of the problem is

   $$\max b^T u$$

   s.t.

   $$A^T u \leq c$$

   $$u \text{ free}$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then

$$A^{-1} = \frac{1}{detA} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$