MATH.APP.220 Multivariable Calculus / Hirvonen

Exam 27.02.2023

No calculators, no written material. A collection of formulas is on the flipside.

1. (a) Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}.$$

   (b) Consider the set T that contains that part of the unit sphere $x^2+y^2+z^2 \leq 1$ where $x \leq 0$, $y \leq 0$ and $z \leq 0$. Write the following integral in spherical coordinates.

$$\iiint_T 1\, dxdydz.$$

   Note: You don't have to evaluate the integral, just change the variables.

2. Find the extreme values of $f(x,y) = x^2y - xy^2 + xy + 1$ in a triangle bounded by the x-axis, the y-axis and the line $y = x + 2$.

3. Consider the functions

$$f(x,y,z) = yz + xz, \qquad G(u,v) = \left(u^2 - v^2, u^2 + v^2, u^2v^2\right).$$

   (a) Find the partial derivatives of $f \circ G$ using the chain rule.
   Note: If you don't remeber how to do it using the chain rule, you can do it without the chain rule, but you will get less points.

   (b) Find all second order partial derivatives of $f \circ G$.

4. A solid is bounded by the paraboloid $z = 2 - x^2 - y^2$ and the cone $z = \sqrt{x^2+y^2}$. Its density is given by

$$\rho(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}.$$

   The solid is cut in half along the xz-plane. Calculate the y-coordinate of the center of mass for one half of the solid. (Choose yourself which half, they are symmetrical.)

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Exam Formula Sheet

1. $T(\mathbf{x}) = F(\mathbf{a}) + F'(\mathbf{a})(\mathbf{x} - \mathbf{a})$

2. $(F \circ G)'(\mathbf{x}) = F'(G(\mathbf{x}))\, G'(\mathbf{x})$

3. $$F'(\mathbf{x}) = \begin{bmatrix} D_1 f_1(\mathbf{x}) & D_2 f_1(\mathbf{x}) & \cdots & D_n f_1(\mathbf{x}) \\ D_1 f_2(\mathbf{x}) & D_2 f_2(\mathbf{x}) & \cdots & D_n f_2(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_m(\mathbf{x}) & D_2 f_m(\mathbf{x}) & \cdots & D_n f_m(\mathbf{x}) \end{bmatrix}$$

4. $D_{\mathbf{e}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{e}$

5. $$\iint_R f(x,y)\, dx dy = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr d\theta$$

6. $$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases} \qquad dx dy dz = \rho^2 \sin\phi\, d\rho d\phi d\theta$$

7. $$m = \iiint_T \rho(x,y,z)\, dV$$
$$\overline{x} = \frac{1}{m}\iiint_T x\rho(x,y,z)\, dV, \quad \overline{y} = \frac{1}{m}\iiint_T y\rho(x,y,z)\, dV, \quad \overline{z} = \frac{1}{m}\iiint_T z\rho(x,y,z)\, dV$$
$$I_z = \iiint_T \left(x^2 + y^2\right) \rho(x,y,z)\, dV$$

8. $$\iint_A f(x,y)\, dx dy = \iint_S f(F(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

9. $$\int_a^b f'(g(x))\, g'(x)\, dx = \Big[f(g(x)) \Big]_a^b$$
$$\int_a^b f'(x)\, g(x)\, dx = \Big[f(x)\, g(x) \Big]_a^b - \int_a^b f(x)\, g'(x)\, dx$$
$$\int_a^b \frac{f'(x)}{f(x)} dx = \Big[\ln|f(x)| \Big]_a^b$$

10. $\sin^2 t = \frac{1}{2}(1 - \cos(2t)), \qquad \cos^2 t = \frac{1}{2}(1 + \cos(2t))$